K-theoretic Quasimap Wall-crossing.

based on joint work with Ming Zhang

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X = smooth projective variety

Enumerative geometry:

moduli of geom. obj. related to X (compactified) $\xrightarrow[\text{intersection theory}]{\text{(virtual)}}$ invariants of X

operators on $H^*(X)$, $K(X)$

$\{$  $\xrightarrow{f} X \}$ $\rightsquigarrow$ Gromov – Witten invariants

$$\|$$

$$\overline{M}_{g,n}(X,\beta) \xrightarrow{ev_i} X$$

evaluation map

moduli of stable maps

$$\langle \gamma_1, \dots, \gamma_n \rangle^X_{g,n,\beta} := \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^{n} ev_i^*(\gamma_i) \in \mathbb{Q}$$

$\gamma_i \in H^*(X)$

virtual fundamental class

$$\langle \alpha_1, \dots, \alpha_n \rangle^X_{g,n,\beta} := \chi\left(\prod_{i=1}^{n} ev_i^*(\alpha_i) \cdot \mathcal{O}^{vir}_{\overline{M}_{g,n}(X,\beta)} \right) \in \mathbb{Z}$$

$\alpha_i \in K^0(X)$

holomorphic Euler char.

virtual structure sheaf

K-theoretic Gromov-Witten theory :

Y = any algebraic scheme or stack

Grothendieck group

coherent sheaves

$K_0(Y) = G(Coh(Y))$ ⇝ homology

vector bundles

$K^0(Y) = G(Vect(Y))$ ⇝ cohomology

- Perfect obstruction theory ⇝ $\mathcal{O}_Y^{vir}$ (Y.-P. Lee)

when Y smooth of expected dimension

$$\mathcal{O}_Y^{vir} = \mathcal{O}_Y$$ — usual structure sheaf

when

$$\begin{array}{c} E \\ \pi \downarrow \uparrow s \\ (s=0) = Y \hookrightarrow W \\ \text{smooth} \end{array}, \quad \mathcal{O}_Y^{vir} = \mathcal{O}_{C_{Y/W}} \overset{L}{\otimes} \mathcal{O}_{s^{-1}(0)}$$

normal cone

Two main differences:

(1) S_n-equivariant theory (Givental)

$$\begin{tikzcd} \overline{M}_{g,n}(X,\beta) \arrow[r, "\tau"] \arrow[dr, "p"'] & {[\overline{M}_{g,n}(X,\beta)/S_n]} \arrow[d, "q"] \\ & \text{point} \end{tikzcd}$$

ordered

unordered

- $p_*\left(\prod_{i=1}^{n} ev_i^*(\gamma)\cdot \mathcal{O}^{vir}_{\overline{M}}\right)$ (virtual)-S_n representation

pulls back

- $q_*\left(\prod_{i=1}^{n} ev_i^*(\gamma)\cdot \mathcal{O}^{vir}_{[\overline{M}/S_n]}\right) = (\qquad)^{S_n}$ — fixed part

(2) $\mathcal{X}$ = orbifold.

conjugation

$$I\mathcal{X} = \bigcup_{x\in X} \left[\mathrm{Aut}(x) / \mathrm{Aut}(x)\right]$$

inertia stack

$$\bar{I}\mathcal{X} = \bigcup_{x\in X} \left[\underset{\substack{\rotatebox{90}{$\in$}\\ g}}{\mathrm{Aut}(x)} \Big/ \left(\mathrm{Aut}(x)/\langle g\rangle\right)\right]$$

rigidified inertia stack

Eg. $\mathcal{X} = \mathbb{P}(2,1)$ 

$I\mathcal{X} =$ 

$\bar{I}\mathcal{X} =$ 

just usual point

$$\begin{tikzcd} \mathcal{C} \supset \Sigma_i \arrow[r, "f"] \arrow[d] & IX \arrow[d] \\ \overline{M}_{g,n}(X,\beta) \arrow[r, "ev_i"] & \bar{I}X \end{tikzcd}$$

i-th marking

gerbe

Cohomology $/\mathbb{Q}$:

$$H^*(\bar{I}X) = H^*(IX) =: H^*_{CR}(X)$$

Chen-Ruan cohomology

For K-theory:

$$K(IX) \neq K(\bar{I}X)$$

Quasimaps

$$X := \left(\sum_{i=0}^{4} x_i^5 = 0\right) \subset \mathbb{P}^4$$

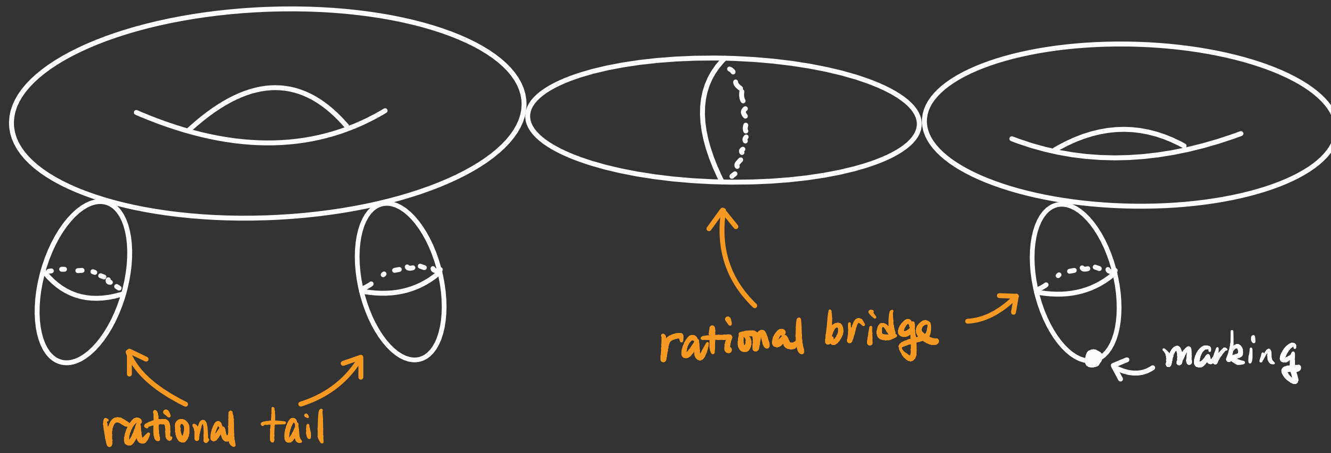
$$\{ C \longrightarrow X \} \xleftrightarrow{1-1} \left\{ \begin{array}{l} L \\ \downarrow \\ C \end{array}, \begin{array}{l} \varphi_0, \ldots, \varphi_4 \in H^0(C, L) \\ \sum \varphi_i^5 = 0 \\ \text{no common zeros} \end{array} \right\}$$

$$X := \left(\sum_{i=0}^{4} x_i^5 = 0\right) \subset \mathbb{P}^4$$

$$\left\{ C \overset{}{\underset{\text{qmap}}{\dashrightarrow}} X \right\} := \left\{ \begin{array}{l} L \\ \downarrow \\ C \end{array} \begin{array}{l} \text{(i) } \varphi_0, \ldots, \varphi_4 \in H^0(C, L) \\ \text{(ii) } \sum \varphi_i^5 = 0 \\ \text{(iii) } (\vec{\varphi} = 0) \text{ discrete, disjoint from nodes \& markings} \end{array} \right\}$$

base locus

proper ?

$C =$ 

<u>Defn</u>: (Ciocan-Fontanine–Kim–Maulik, '14)

A quasi-map is ε-stable $(\varepsilon \in \mathbb{Q}_{>0})$ if

- Every rational bridge has degree > 0
- Every rational tail has degree $> \frac{1}{\varepsilon}$
- Every base point x has length $\le \frac{1}{\varepsilon}$

$$\text{length} = \min_i \{\operatorname{ord}_x \varphi_i\}$$

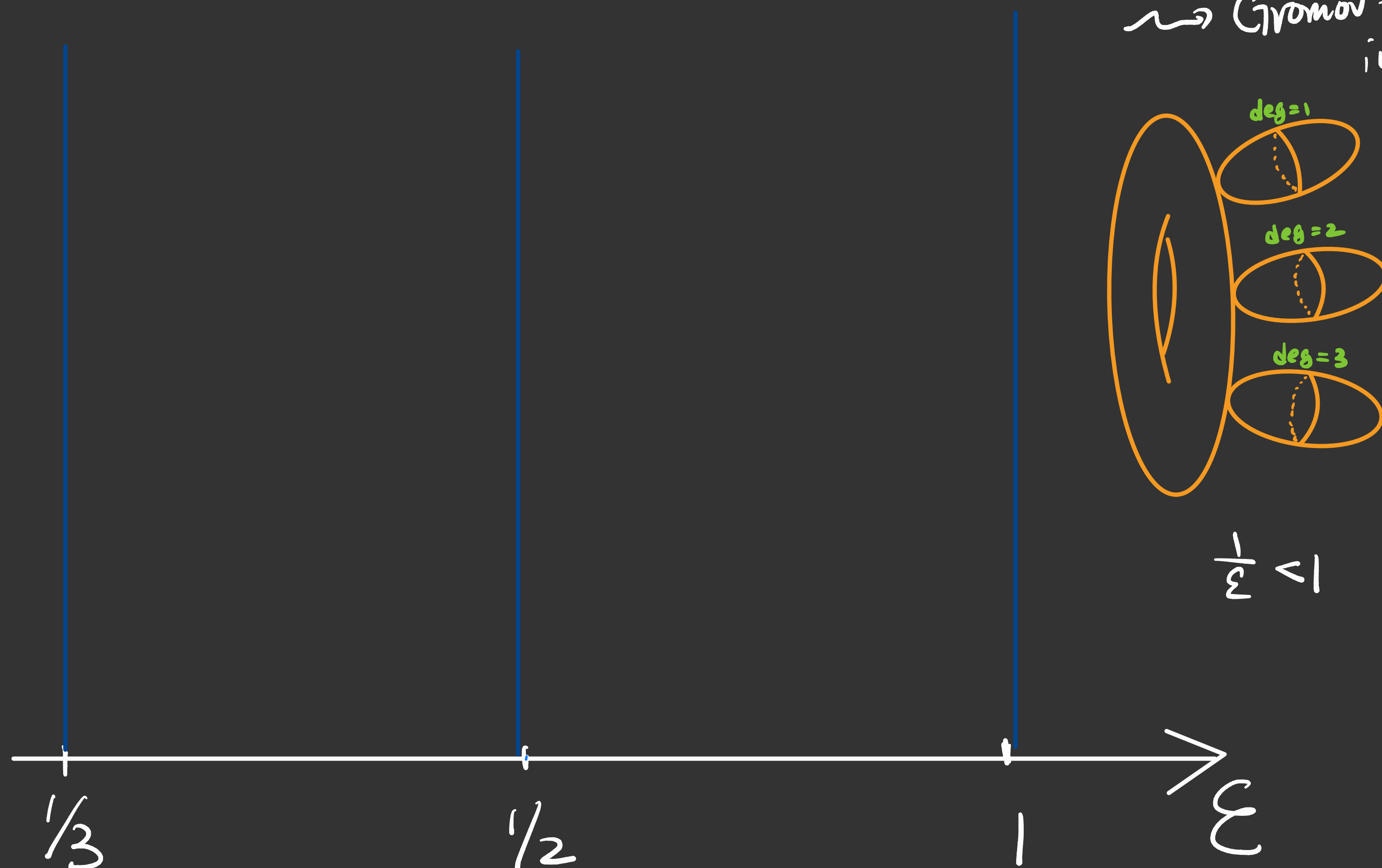
+ 2 special cases:

deg > 0

deg $> \frac{2}{\varepsilon}$

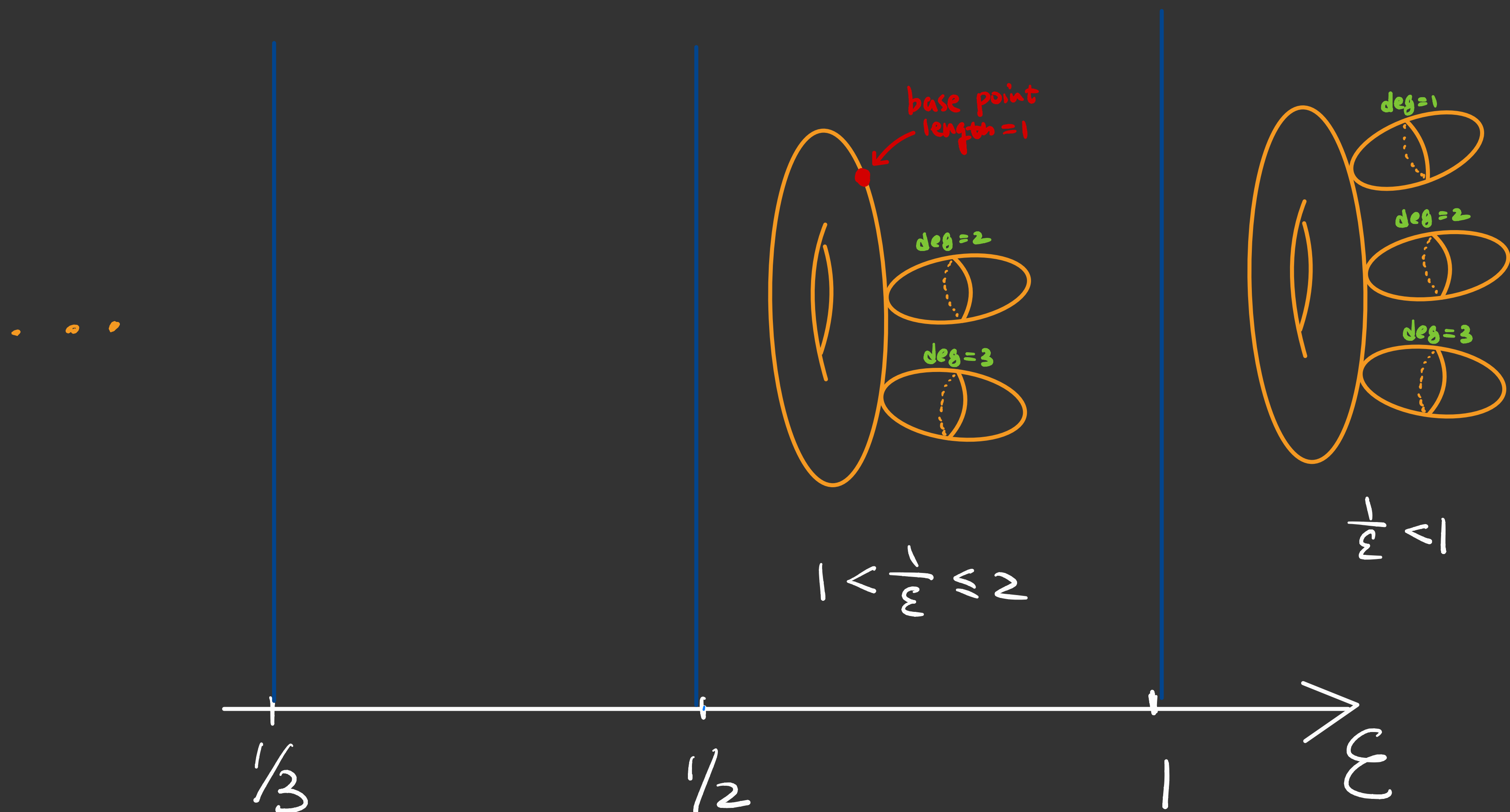
Wall-and-chamber structure:

stable maps
$\rightsquigarrow$ Gromov-Witten invariants



$\frac{1}{\varepsilon} < 1$

Wall-and-chamber structure:



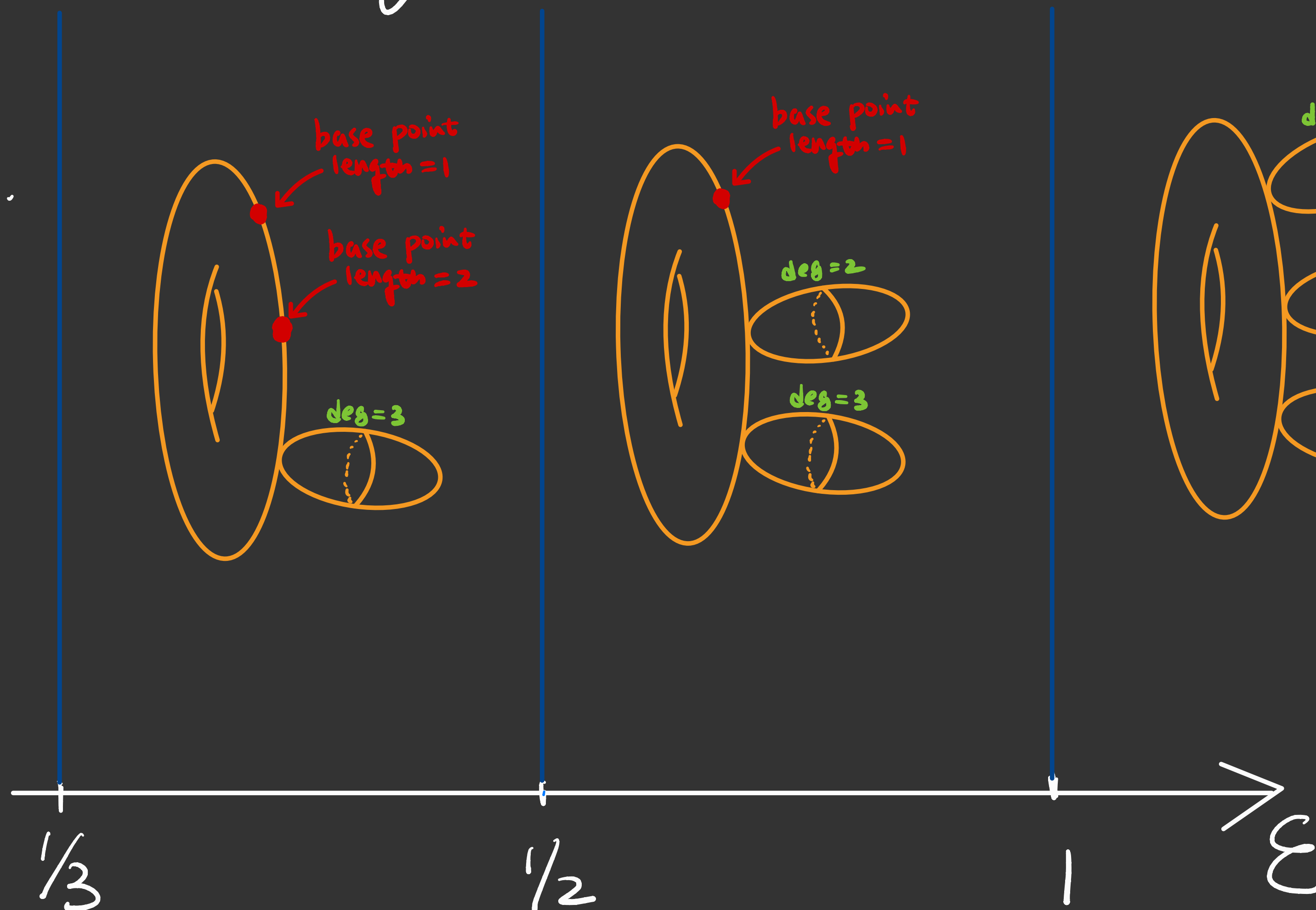
Wall-and-chamber structure:

$\varepsilon \to 0^+ \longrightarrow$ stability condition

no rational tails at all.

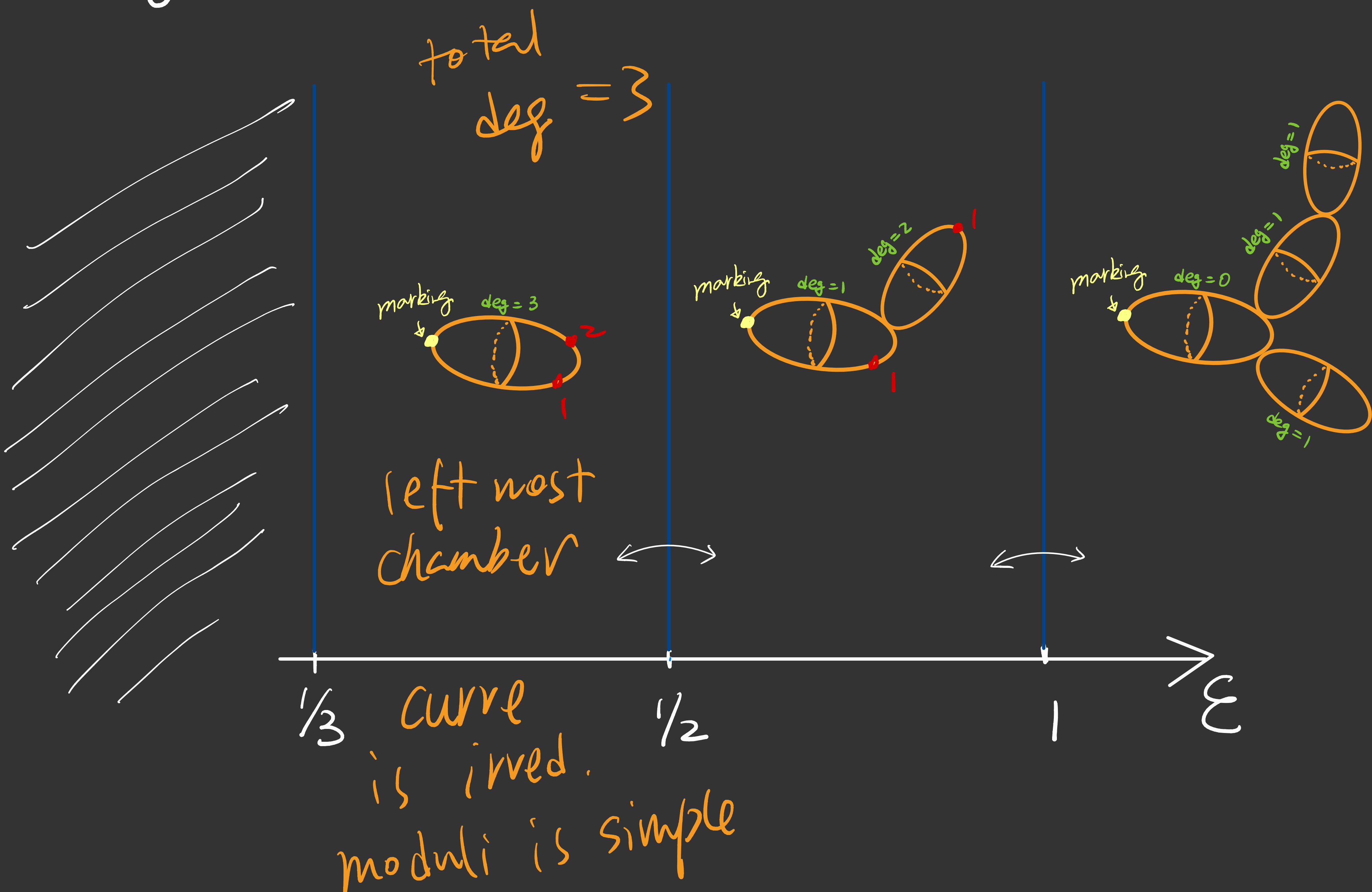
. . .

B-model amplitude



∞-many walls, but for fixed deg,
$1, \ 1/2, \ \ldots, \ 1/d$

$g=0\ ,\ n=1$


total deg = 3
marking
deg = 3
2
1
marking
deg = 1
deg = 2
1
1
marking
deg = 0
deg = 1
deg = 1
deg = 1
left most chamber
1/3
1/2
1
ε
curve is irred.
moduli is simple

More general target space:

affine, lci sing.

$W^s(\theta) = W^{ss}(\theta)$, = smooth.

reductive group

$$X = W /\!/_{\theta} G \subset [W/G]$$

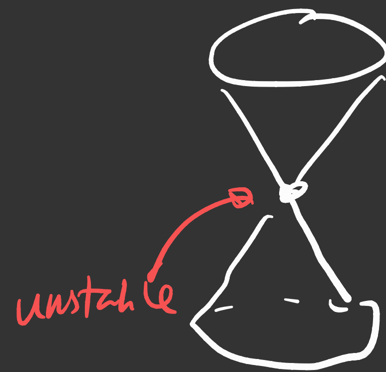
char of G

Artin stack

$$\{C \underset{\text{qmap}}{\dashrightarrow} X\} := \left\{ \begin{array}{l} C \xrightarrow{u} [W/G] \\ \text{s.t } u^{-1}([W^{us}(\theta)/G]) \text{ discrete and disjoint from markings and nodes} \end{array} \right\}$$

!! base locus.

$$C \to [W/G] \iff \begin{array}{c} G-P \\ \downarrow \\ C \end{array} + \begin{array}{c} P \times_G W \\ \downarrow \uparrow \sigma \\ C \end{array}$$

Quintic $X =$ 

affine cone $\left(\sum x_i^5 = 0\right) \subseteq \mathbb{C}^5$ $/\!/ \ \mathbb{C}^*$

$\mathbb{C}^*$-principal bundle $P \longleftrightarrow$ line bundle $\begin{array}{c} L \\ \downarrow \\ C \end{array}$

$$\begin{array}{ccc} P \times_G W & = \left(\sum x_i^5 = 0\right) \subset & \mathrm{Tot}\left(L^{\oplus 5}\right) \\ \downarrow & & \downarrow \\ C & & C \end{array}$$

<u>More examples:</u>

(Complete intersections in) Grassmannian, flag varieties, quiver varieties, A very special case:

$$X = pt = \mathbb{C} /\!/ \mathbb{C}^*$$

$$\left\{ C \underset{\text{qmap}}{\dashrightarrow} X \right\} \longleftrightarrow \left\{ \begin{array}{l} d \text{ markings of} \\ \text{weight } \varepsilon \end{array} \right\}$$

$\deg = d$

ε-stable

Hassett's moduli of weighted pointed curves.

Thm (Ciocan-Fontanine — Kim — Maulik, Cheong — Ciocan-Fontanine — Kim (orbifold))

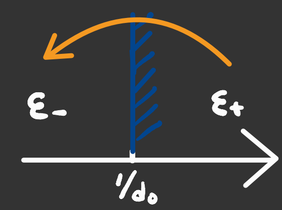
$Q^{\varepsilon}_{g,n}(X,\beta) := \{\varepsilon\text{-stable quasimaps to } X\}$

is a proper DM stack / $\mathbb{C}$,

with a perfect obstruction theory.

$\rightsquigarrow [Q^{\varepsilon}_{g,n}(X,\beta)]^{vir}$ and $\mathcal{O}^{vir}_{Q^{\varepsilon}_{g,n}(X,\beta)}$

Cohomological Wall-crossing formula.



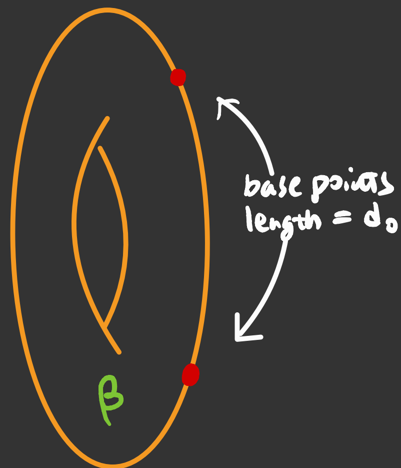
Conjecture: (Ciocan-Fontanine — Kim)

$$[Q^{\varepsilon_-}_{g,n}(X,\beta)]^{vir} - [Q^{\varepsilon_+}_{g,n}(X,\beta)]^{vir}$$

comb. of $H^*(X)$ & ψ-classes

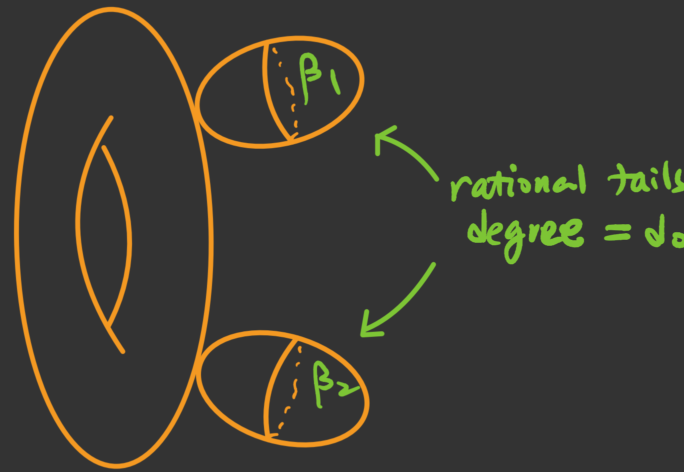
$$\text{"}=\text{"} \sum_{k\geq 1} \sum_{\vec{\beta}} \frac{1}{k!} \prod_{i=1}^{k} ev_{n+i}^{*} \mu_{\beta_i}(z)\Big|_{z=-\psi_{n+i}} \cap [Q^{\varepsilon_+}_{g,n+k}(X,\beta')]^{vir}$$

where $\vec{\beta} = (\beta', \beta_1, \cdots, \beta_k)$. $\beta = \beta' + \beta_1 + \cdots + \beta_k$. $\deg(\beta_i) = d_0$

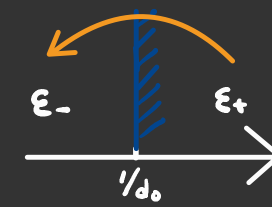


$Q^{\varepsilon_-}_{g,n}(X,\beta)$

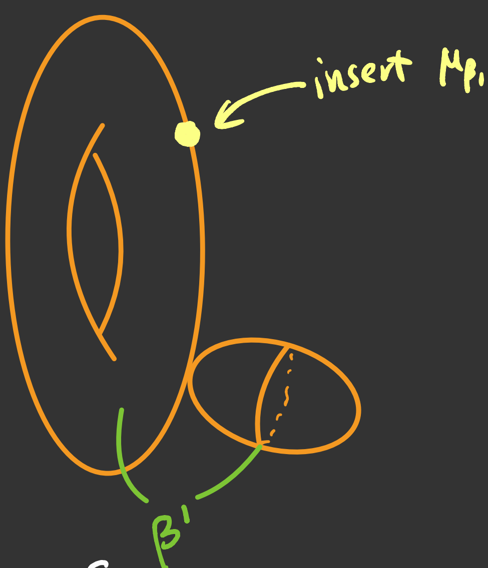
$-$



$Q^{\varepsilon_+}_{g,n}(X,\beta)$



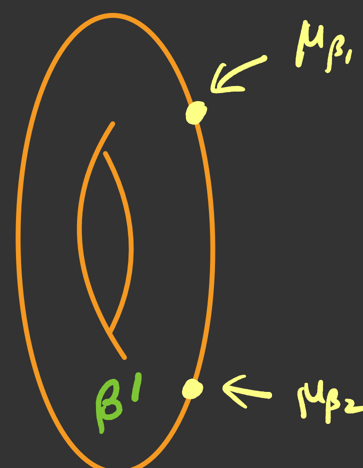
$=$



$Q^{\varepsilon_+}_{g,n+1}(X,\beta')$

$\beta = \beta' + \beta_1$

$+ \frac{1}{2!}$



$Q^{\varepsilon_+}_{g,n+2}(X,\beta')$

$\beta = \beta' + \beta_1 + \beta_2$

$+ \cdots$

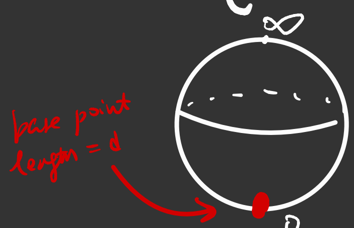
μ_{β_i} = truncation of the I-function.

Consider:

$$\left\{ \mathbb{C}^* \curvearrowright \mathbb{P}^1 \overset{[f_0,\dots,f_4]}{\dashrightarrow} X = \text{quintic} \right\}$$

Fixed locus

$$F_\star := \left\{ [a_0 x^d, \dots, a_4 x^d] \,\middle|\, [a_0,\dots,a_4] \in X \right\} \cong X$$



$$\overset{\deg = d}{\dashrightarrow} X$$

Localization residue $\rightsquigarrow$ I-function.

$$\frac{[F_\star]}{e^{\mathbb{C}^*}(N_{F_\star})} \in H^*_{\mathbb{C}^*}(X)_{loc} \cong H^*(X)[z, z^{-1}]$$

equivariant parameter
generator of $H^*(\mathbb{P}^\infty)$

The conjecture was proved by

- Ciocan-Fontanine-Kim

  for c.i. in (products of) $\mathbb{P}^N$ all genera.

  spaces with nice torus action $g=0$, $g>0$ (with semipositive condition.)

- Cheong-CFK

  spaces with nice torus action, $g=0$

- Clader-Janda-Ruan

  c.i. in (products of) $\mathbb{P}^N$ all g.

- J. Wang

  c.i. in toric orbifolds, $g=0$.

Thm (Z–, 2021)

- The conjectured wall-crossing formula is true for all targets in all genera. (including the orbifold case).
- $(g=0, n=1)$ the left most chamber can be computed as an explicit truncation of the I-function.

K-theoretic version:

S_n-equivariance

__Thm:__ (M. Zhang – Z)

$$\mathcal{O}^{vir}_{Q^{\varepsilon_-}_{g,n}(X,\beta)} - \mathcal{O}^{vir}_{Q^{\varepsilon_+}_{g,n}(X,\beta)}$$

$$\text{“}=\text{''} \sum_{k\geq 1} \sum_{\vec{\beta}} \prod_{i=1}^{k} ev_{n+i}^{*}\,\mu_{\beta_i}(L) \cap \mathcal{O}^{vir}_{[Q^{\varepsilon_+}_{g,n+k}(X,\beta')/S_k]}$$

cotangent line class at marking

where $\vec{\beta} = (\beta', \beta_1, \cdots, \beta_k)$. $\beta = \beta' + \beta_1 + \cdots + \beta_k$. $\deg(\beta_i) = d_0$

Definition of the $\mu_{\beta_i}(L)$ or I is different in the orbifold case.

$$
\begin{tikzcd}
\{\mathbb{P}(r,1) \arrow[r, dashed] & X\} \\
\cup & \\
F_{\star} & \\
\Sigma \arrow[u] \arrow[r, "f"] & IX
\end{tikzcd}
\qquad \sigma : F_{\star} \to \Sigma
$$

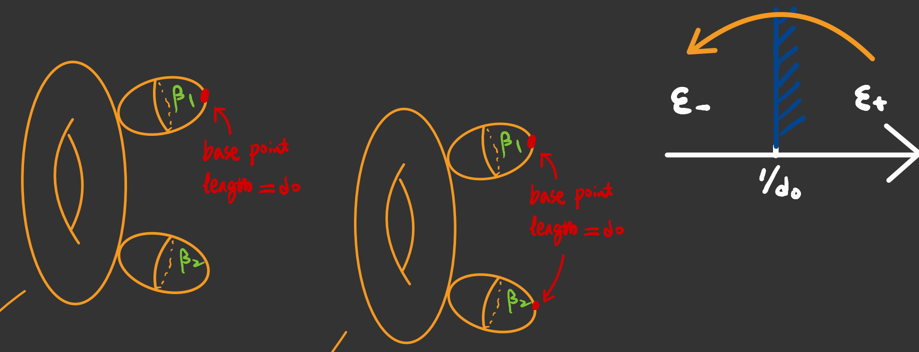
orbifold marking

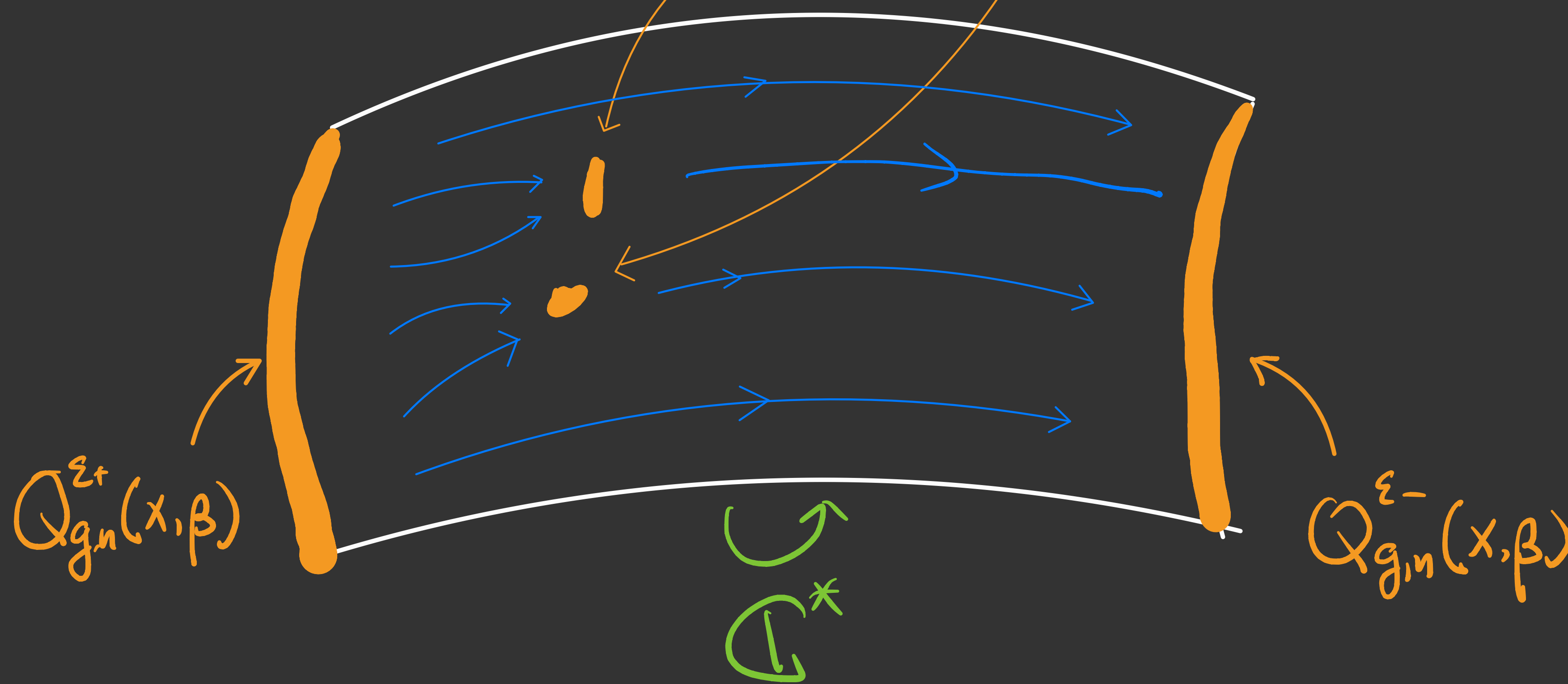
$$\mathrm{Cont}_{F_{\star}} = (ev_{\star})_{*}\,\sigma_{*}\left(\frac{\mathcal{O}^{\mathrm{vir}}_{F_{\star}}}{e^{\mathbb{C}^*}(N_{F_{\star}})}\right) \in K(X)[q^{1/r}; \mathcal{L}$$

universal line bundle over $\mathbb{CP}^{\infty}$.

σ is not equivariant!

Idea of proof:





Master space.

- properness
- preserve p.o.t.

- Allows base points of length d_0
- Disallows rat'l tails of degree d_0

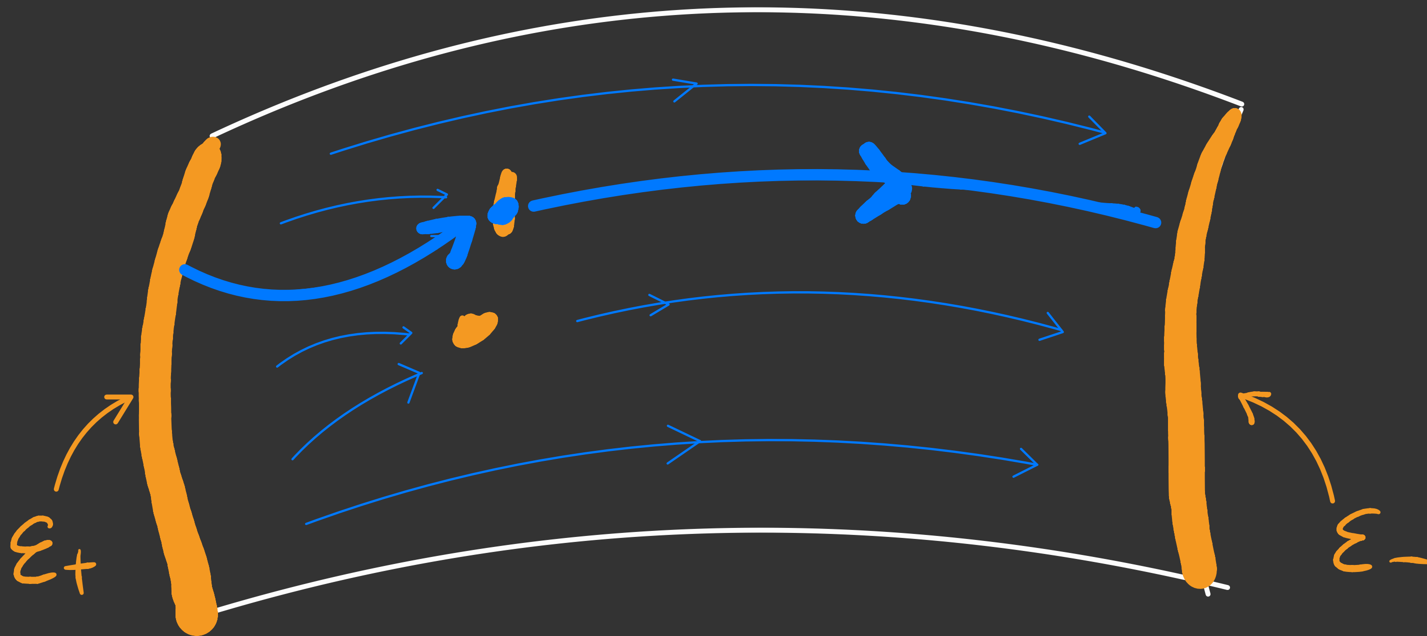
ε_- $\qquad$ ε_+

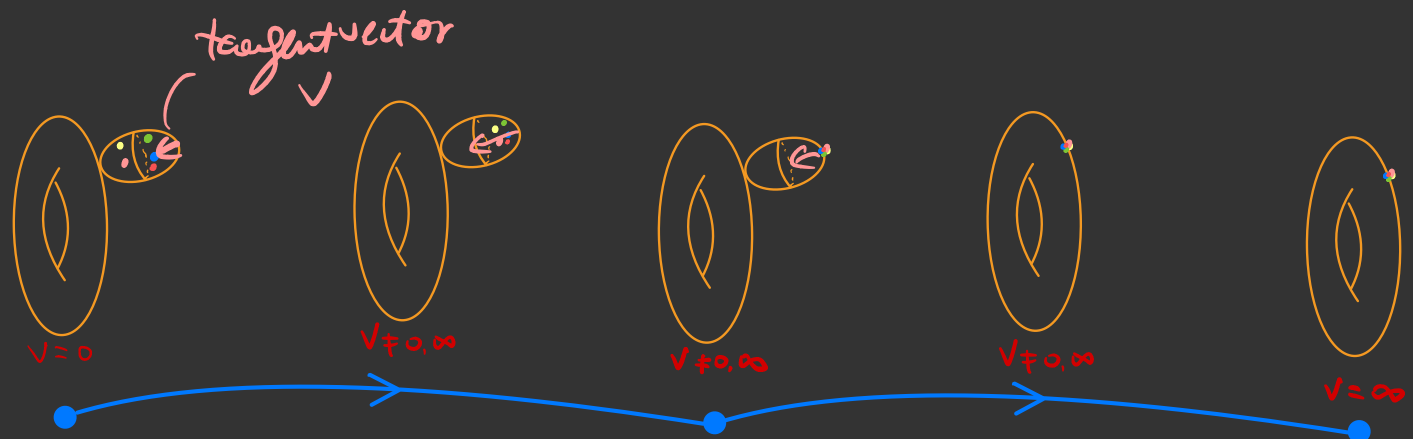
- Disallows base points of length d_0
- Allows rat'l tails of degree d_0

$$\tfrac{1}{d_0} = \varepsilon_0$$

$\mathbb{C}^*$-action fixes the q-map just as in the defn of the I-function!!

Aut() $\supset \mathbb{C}^*$

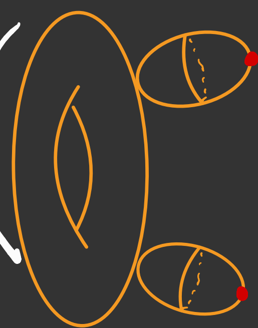




$\mathcal{E}_+$ $\mathcal{E}_-$

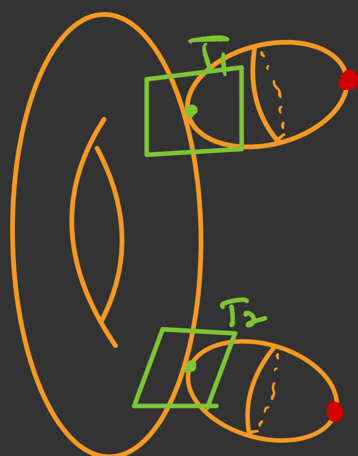
stability condition :

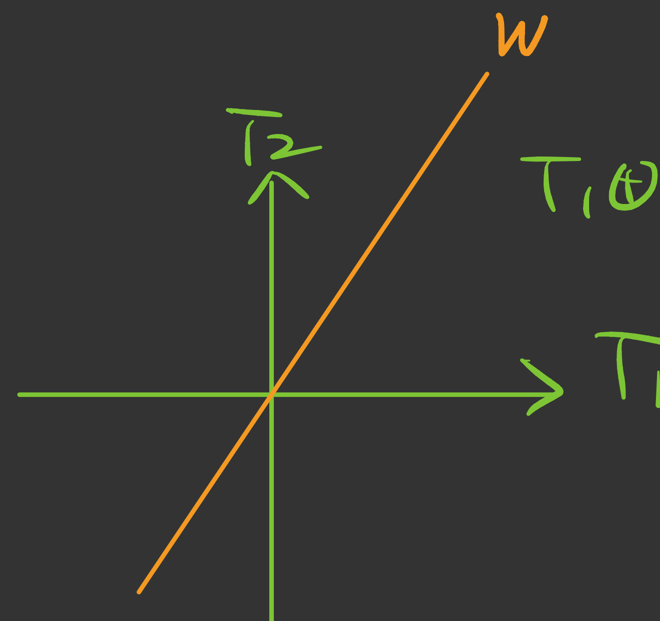
- when $v=\infty$ 
- . . .

Aut() $\supset \mathbb{C}^* \times \mathbb{C}^*$

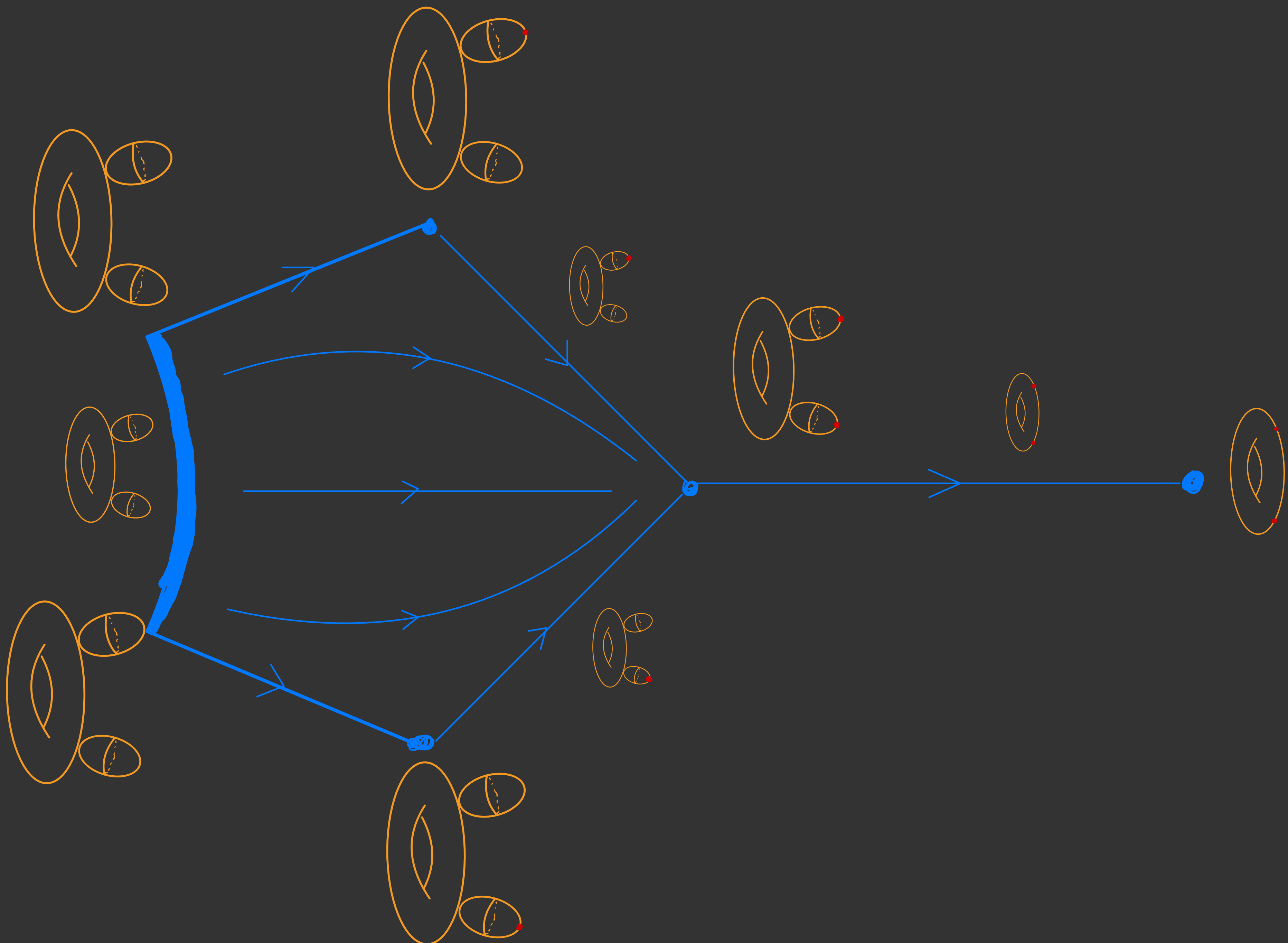
How to break symmetric to choose a  to reparametrize?

"Entanglement": entangled tail.



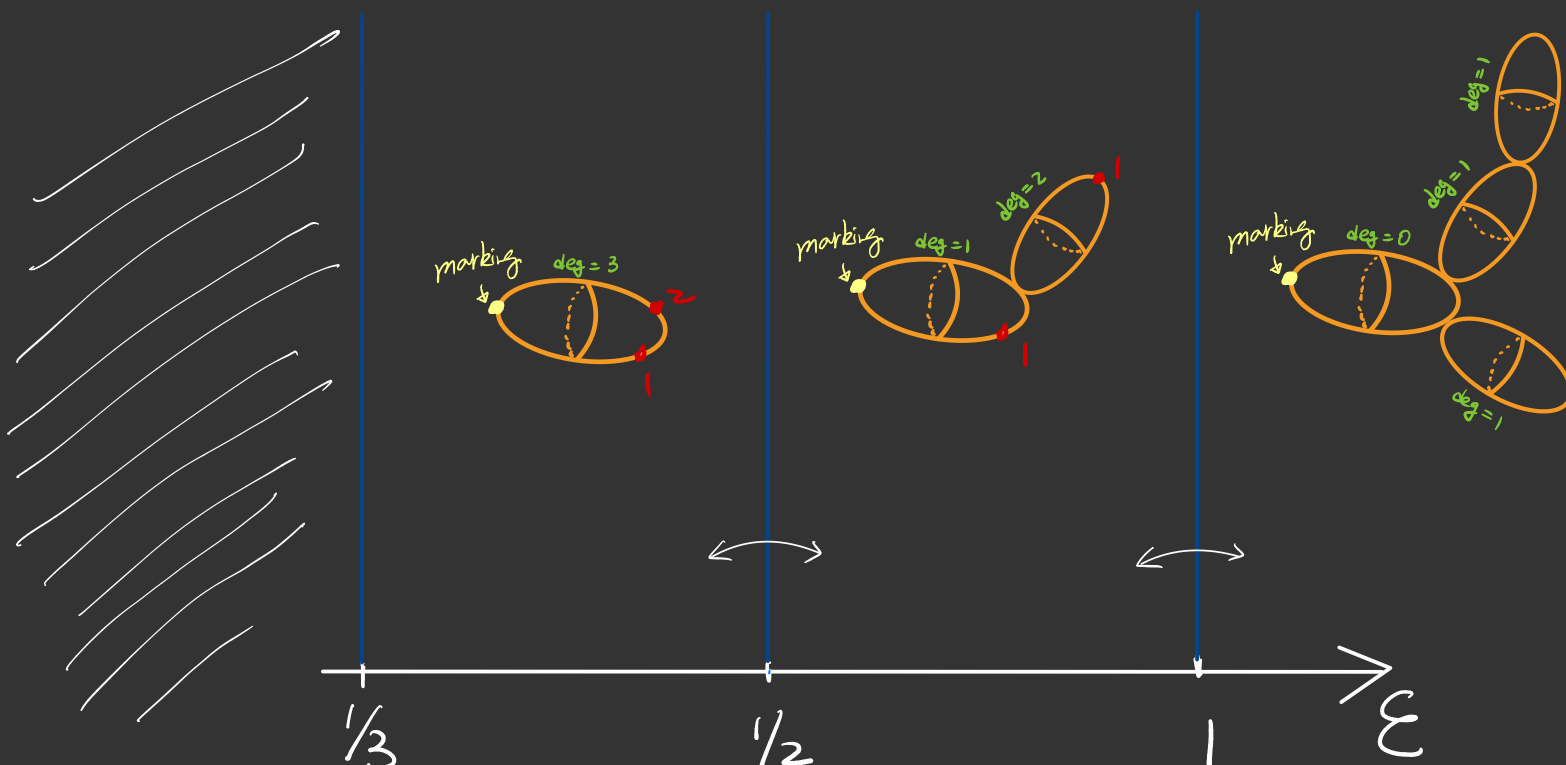


$T_1 \oplus T_2 \cong \mathbb{C}^2$

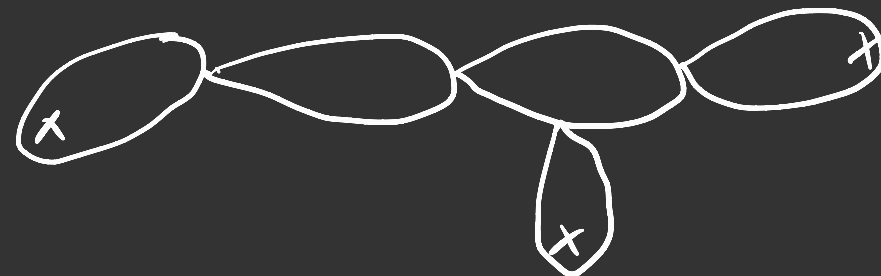


Towards computing the Quantum K-ring for toric complete intersections

Recall: $g=0$, $n=1$



Trouble: the correction term has many markings.

$\varepsilon \to 0^+$ 

For cohomological GW, we have

"divisor equation", $\rightsquigarrow$ solution in CY case.

Possible solutions: (Work in progress)

① Quantum difference equation for I-function.

② Use light marking.

Thank you!